REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. III

BY

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The main object of this paper is to define the character of an irreducible quasi-simple(1) representation π of a connected semisimple Lie group G on a Hilbert space $\mathfrak{H}$. This will be done as follows. Let $C_c^\infty(G)$ be the class of all functions on G which are indefinitely differentiable and which vanish outside a compact set. For any $f \in C_c^\infty(G)$ we consider the operator $\int f(x)\pi(x)dx$ where dx is the Haar measure on G. It turns out that this operator has a trace (which we denote by $T_\pi(f)$) and the mapping $T_\pi: f \rightarrow T_\pi(f)$ $(f \in C_c^\infty(G))$ is a *distribution* in the sense of L. Schwartz [9] such that $T_\pi(f_a) = T_\pi(f)$ where $f_a(x) = f(axa^{-1})$ $(a, x \in G)$. This distribution is defined to be the character of π. We shall see that two such representations π_1, π_2 are infinitesimally equivalent (see [6, §9]) if and only if they have the same character. Therefore in particular a unitary irreducible representation is determined within unitary equivalence by its character (cf. Theorem 8 of [6]).

In the last section we give a simple proof of a formula for "spherical functions" on a *complex* semisimple group. This formula was obtained by Gelfand and Naimark [1; 2] in some special cases by direct computation.

1. **Some preliminary results.** We keep to the notation of our two earlier papers [6, 7] on the same subject. G is a connected, simply connected, semisimple Lie group and $\mathfrak{g}_0$ is its Lie algebra over the field R of real numbers. $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$ and $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{c}$, $\mathfrak{k}'$, and $\mathfrak{m}$ are defined as in [6, §2] and [7, §2]. K, K', and D are the analytic subgroups of G corresponding to $\mathfrak{k}_0$, $\mathfrak{k}_0' = \mathfrak{k}' \cap \mathfrak{g}_0$ and $\mathfrak{c}_0 = \mathfrak{c} \cap \mathfrak{g}_0$ respectively. Let Z be the center of G and $\mathfrak{Z}$ the center of the enveloping algebra $\mathfrak{B}$ of $\mathfrak{g}$. If π is a representation of G on a Banach space we shall say that π is quasi-simple if it maps the elements of $D \cap Z$ and $\mathfrak{Z}$ into scalar multiples of the unit operator (see [6, §10]).

Let π be a quasi-simple irreducible representation of G on a Banach space $\mathfrak{H}$. We denote by Ω the set of all equivalence classes of finite-dimensional simple representations of K. Let $\mathfrak{H}_\mathfrak{D}$ $(\mathfrak{D} \in \Omega)$ denote the set of all elements in $\mathfrak{H}$ which transform under $\pi(K)$ according to $\mathfrak{D}$. We know (see [6, Lemma 33]) that $\dim \mathfrak{H}_\mathfrak{D} < \infty$. Let $E_\mathfrak{D}$ denote the canonical projection of $\mathfrak{H}$ on $\mathfrak{H}_\mathfrak{D}$ (see [6, §9]). For any $x \in G$ consider the operator $E_\mathfrak{D}\pi(x)E_\mathfrak{D}$. It maps $\mathfrak{H}_\mathfrak{D}$ into itself and $\mathfrak{H}_{\mathfrak{D}'}$ into $\{0\}$ $(\mathfrak{D}' \neq \mathfrak{D})$. Let sp $(E_\mathfrak{D}\pi(x)E_\mathfrak{D})$ denote the trace of the restriction of $E_\mathfrak{D}\pi(x)E_\mathfrak{D}$ on $\mathfrak{H}_\mathfrak{D}$. Since $\dim \mathfrak{H}_\mathfrak{D} < \infty$ this trace is well defined. Now any given linear function α on $\mathfrak{H}_\mathfrak{D}$ may be extended to a continuous linear function on $\mathfrak{H}$ by setting $\alpha(\psi) = \alpha(E_\mathfrak{D}\psi)$ $(\psi \in \mathfrak{H})$. In particular if ψ_i,

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(1) See [6, §10] for the definition of a quasi-simple representation.

$1 \leqq i \leqq r$, is a base for $\mathfrak{H}_{\mathfrak{D}}$ and $\tilde{\psi}_i$ is the linear function on $\mathfrak{H}_{\mathfrak{D}}$ which takes the value 1 at ψ_i and zero at ψ_j $(j \neq i,\ 1 \leqq i,j \leqq r)$ we may extend $\tilde{\psi}_i$ on $\mathfrak{H}$ in the above fashion. Then it is clear that

$$\phi_{\mathfrak{D}}^{\pi}(x) = \operatorname{sp} E_{\mathfrak{D}}\pi(x)E_{\mathfrak{D}} = \sum_{i=1}^{r} (\tilde{\psi}_i, \pi(x)\psi_i)$$

in the notation of [6, §10]. Hence it follows from Lemmas 19 and 34 of [6] that $\phi_{\mathfrak{D}}^{\pi}$ is an analytic function on G. $X_1, \cdots, X_n$ being a base for $\mathfrak{g}_0$ over R, set $X(t) = t_1X_1 + \cdots + t_nX_n$ $(t_j \in R)$. Then we know (cf. Theorem 2 and Lemma 34 of [6]) that if $|t| = \max_j |t_j|$ is sufficiently small we get the convergent expansions

$$\pi(\exp X(t))\psi_i = \sum_{m \geqq 0} \frac{1}{m!}\pi((X(t))^m)\psi_i, \qquad 1 \leqq i \leqq r.$$

From this it follows immediately that if z is any element in $\mathfrak{B}$ the value of $\sum_{1 \leqq i \leqq r} (\tilde{\psi}_i, \pi(z)\psi_i)$ can be obtained in terms of the various partial derivatives of $\phi_{\mathfrak{D}}^{\pi}$ $(\exp X(t))$ with respect to (t) at $t_1 = t_2 = \cdots = t_n = 0$. Let σ be the representation of $\mathfrak{A} = \mathfrak{Q}\mathfrak{X}$ (see [7, §2] for notation) on $\mathfrak{H}_{\mathfrak{D}}$ defined under π. Then the knowledge of the function $\phi_{\mathfrak{D}}^{\pi}$ determines in particular $\operatorname{sp} \sigma(z)$ for any $z \in \mathfrak{A}$. Now we know (see Theorem 5 of [6]) that π defines a quasi-simple(2) irreducible representation of $\mathfrak{B}$ on $\mathfrak{H}^{(0)} = \sum_{\mathfrak{D}' \in \Omega} \mathfrak{H}_{\mathfrak{D}'}$ and therefore σ is irreducible (see Corollary 2 to Theorem 2 of [7]). On the other hand a finite-dimensional simple representation of an associative algebra is completely determined within equivalence by its trace (see Lemma 16 of [7]). Hence in view of Theorem 2 of [7] we can conclude that the function $\phi_{\mathfrak{D}}^{\pi}$ determines the representation of $\mathfrak{B}$ on $\mathfrak{H}^{(0)}$ up to equivalence and therefore the representation π of G up to infinitesimal equivalence. This result may be stated in a slightly more general form as follows.

THEOREM(3) 1. *Let $\pi_1, \cdots, \pi_r$ be a finite set of quasi-simple irreducible representations of G on Banach spaces. Suppose no two of them are infinitesimally equivalent. Then all the nonzero functions in the set $\phi_{\mathfrak{D}_1}^{\pi_1}, \cdots, \phi_{\mathfrak{D}_r}^{\pi_r}$ $(\mathfrak{D}_i \in \Omega,\ 1 \leqq i \leqq r)$ are linearly independent.*

Let C be the field of complex numbers. If our assertion is false we may suppose that $c_1\phi_{\mathfrak{D}_1}^{\pi_1} + \cdots + c_s\phi_{\mathfrak{D}_s}^{\pi_s} = 0$ where $c_j\phi_{\mathfrak{D}_j}^{\pi_j} \neq 0$, $1 \leqq j \leqq s$ $(c_j \in C)$. Let $\mathfrak{H}_i$ be the representation space of π_i. Consider the representation σ_i of $\mathfrak{A}$ on $\mathfrak{H}_{i,\mathfrak{D}_i}$ $(1 \leqq i \leqq s)$ induced under π_i. Then $c_1 \operatorname{sp} \sigma_1(a) + \cdots + c_s \operatorname{sp} \sigma_s(a) = 0$ for all $a \in \mathfrak{A}$. Since $c_j\phi_{\mathfrak{D}_j}^{\pi_j} \neq 0$, $\mathfrak{D}_j$ occurs in π_j. Moreover π_j, π_k $(j \neq k)$ are not infinitesimally equivalent $(1 \leqq j,\ k \leqq s)$. Hence it follows from Corollary 2 to Theorem 2 of [7] that the representations $\sigma_1, \cdots, \sigma_s$ are irreducible and no

(2) See [7, end of §2] for the definition of quasi-simplicity in this case.

(3) Cf. Theorem 7 of [8(a)] and Theorem 2 of [8(b)].

two of them are equivalent. This however gives a contradiction with Lemma 16 of [7]. So the theorem is proved.

We recall that for two irreducible *unitary* representations on Hilbert spaces the notions of infinitesimal equivalence and ordinary equivalence are the same (see Theorem 8 of [6]). Hence if π_1, π_2 are two such representations which are not equivalent, the corresponding functions $\phi_{\mathfrak{D}_1}^{\pi_1}$, $\phi_{\mathfrak{D}_2}^{\pi_2}$ are always distinct unless they are both zero.

Theorem 5 of [7] can now be rephrased in terms of the function $\phi_{\mathfrak{D}}^{\pi}$ as follows:

THEOREM(4) 2. *Let π be a quasi-simple irreducible representation of G on a Banach space $\mathfrak{H}$. Suppose $\mathfrak{D}_0$ is a class in Ω occurring in π such that $d(\mathfrak{D}_0)=1$. Then* $\dim \mathfrak{H}_{\mathfrak{D}_0}=1$ *and it is possible to choose linear functions Λ and μ on $\mathfrak{h}$ and $\mathfrak{c}$ respectively such that*

$$\phi_{\mathfrak{D}_0}^{\pi}(x) = \int_{K^*} e^{\mu(\Gamma(x,u^*))} e^{\Lambda(H(x,u^*))} du^* \qquad (x \in G)$$

and the infinitesimal character of π is χ_Λ.

Some properties of the function $\phi_{\mathfrak{D}}^{\pi}$ have been studied by R. Godement [3] (see also [1; 2]).

We shall now state a few immediate consequences of the results proved in [6].

THEOREM(5) 3. *Let χ be a homomorphism of $\mathfrak{Z}$ into C and $\mathfrak{D}_0$ a class in Ω. Then apart from infinitesimal equivalence there exist only a finite number of irreducible quasi-simple representations π of G which have the infinitesimal character χ and such that $\mathfrak{D}_0$ occurs in π.*

This follows from Theorem 2 of [7]. Similarly the following result is obtained from Theorem 3 of [8].

THEOREM 4. *Let π be a quasi-simple irreducible representation of G on a Banach space $\mathfrak{H}$. Then there exists an integer N such that*

$$\dim \mathfrak{H}_{\mathfrak{D}} \leqq N(d(\mathfrak{D}))^2$$

for all $\mathfrak{D} \in \Omega$.

2. **Trace of an operator.** Let $\{c_\alpha\}_{\alpha \in J}$ be an indexed set of complex numbers. We define the convergence of the series $\sum_{\alpha \in J} c_\alpha$ and its sum in the usual manner (see §5 of [6]). Let A be a bounded operator on a Hilbert space $\mathfrak{H}$ and let $\{\psi_\alpha\}_{\alpha \in J}$ be an orthonormal base for $\mathfrak{H}$. We say that A has a trace

(4) Cf. Theorem 3 of [8(b)]. Our notation is the same as that of Theorem 5 of [7].

(5) Cf. Theorem 6 of [8(a)].

(or A is of the trace class) if for every such base the series[6] $\sum_{\alpha\in J}(\psi_\alpha, A\psi_\alpha)$ converges to a sum which is independent of the choice of the base. The value of this sum is called the trace of A and we shall denote it by sp A.

LEMMA 1. *Let $\{\psi_\alpha\}_{\alpha\in J}$ be an orthonormal base for a Hilbert space $\mathfrak{H}$ and T a bounded operator such that $\sum_{\alpha,\beta\in J}|t_{\alpha\beta}|<\infty$ where $t_{\alpha\beta}=(\psi_\alpha, T\psi_\beta)$. Then if A and B are any bounded operators on $\mathfrak{H}$, ATB, BAT, TBA are all of the trace class and*

$$\mathrm{sp}\,(ATB) = \mathrm{sp}\,(BAT) = \mathrm{sp}\,(TBA).$$

Put $a_{\alpha\beta}=(\psi_\alpha, A\psi_\beta)$, $b_{\alpha\beta}=(\psi_\alpha, B\psi_\beta)$ $(\alpha, \beta\in J)$ and consider the series $\sum_{\alpha,\beta,\gamma\in J}|a_{\alpha\beta}t_{\beta\gamma}b_{\gamma\alpha}|$. Then[7]

$$\sum_\alpha |a_{\alpha\beta}t_{\beta\gamma}b_{\gamma\alpha}| = |t_{\beta\gamma}|\sum_\alpha |a_{\alpha\beta}b_{\gamma\alpha}|$$

$$\leqq |t_{\beta\gamma}|\left(\sum_\alpha |a_{\alpha\beta}|^2\right)^{1/2}\left(\sum_\alpha |b_{\gamma\alpha}|^2\right)^{1/2} \leqq |t_{\beta\gamma}|\,|A|\,|B|$$

since

$$\sum_\alpha |a_{\alpha\beta}|^2 = \sum_\alpha |(\psi_\alpha, A\psi_\beta)|^2 = |A\psi_\beta|^2 \leqq |A|^2$$

and similarly for B. Hence

$$\sum_{\alpha,\beta,\gamma} |a_{\alpha\beta}t_{\beta\gamma}b_{\gamma\alpha}| \leqq |A|\,|B|\sum_{\beta,\gamma}|t_{\beta\gamma}| < \infty.$$

This proves that the series $\sum_{\alpha,\beta,\gamma} a_{\alpha\beta}t_{\beta\gamma}b_{\gamma\alpha}$ is absolutely convergent and so it follows in the usual way that

$$\sum_\alpha (\psi_\alpha, ATB\psi_\alpha) = \sum_\alpha (\psi_\alpha, TBA\psi_\alpha) = \sum_\alpha (\psi_\alpha, BAT\psi_\alpha).$$

Now let U be a unitary transformation on $\mathfrak{H}$. Consider $U^{-1}ATBU$. Since $U^{-1}A$ and BU are bounded operators, we can conclude from the above result that

$$\begin{aligned}\sum_\alpha (U\psi_\alpha, ATBU\psi_\alpha) &= \sum_\alpha (\psi_\alpha, U^{-1}ATBU\psi_\alpha)\\ &= \sum_\alpha (\psi_\alpha, TBUU^{-1}A\psi_\alpha) = \sum_\alpha (\psi_\alpha, TBA\psi_\alpha)\\ &= \sum_\alpha (\psi_\alpha, ATB\psi_\alpha).\end{aligned}$$

Since every orthonormal base in $\mathfrak{H}$ is related to the base $\{\psi_\alpha\}_{\alpha\in J}$ by a unitary transformation, this proves that ATB is of the trace class. Since BA is a

(6) As usual we denote by (ϕ, ψ) the scalar product of ϕ and ψ in $\mathfrak{H}$.

(7) For any bounded operator Q we put $|Q| = \sup_{|\psi|\leqq 1}|Q\psi|$.

bounded operator it follows from this result that BAT and TBA are also of the trace class. Hence in view of the above equalities we conclude that sp ATB = sp BAT = sp TBA.

COROLLARY. *If T satisfies the conditions of the above lemma and if A is a regular operator, then T and ATA^{-1} are both of the trace class and* sp ATA^{-1} = sp T.

3. **An auxiliary lemma.** In order to prove that certain given operators are of the trace class we shall frequently need the following result.

LEMMA 2. *Let $\mathfrak{l}$ be a semisimple Lie algebra over C of rank l. Then the series $\sum_{\mathfrak{D}} d(\mathfrak{D})^{-(l+1)}$ is convergent. Here $\mathfrak{D}$ runs over all equivalence classes of finite-dimensional simple representations of $\mathfrak{l}$ and $d(\mathfrak{D})$ is the degree of any representation in $\mathfrak{D}$.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$. Choose a fundamental system of roots and let $\Lambda_1, \cdots, \Lambda_l$ be a fundamental set of dominant integral functions on $\mathfrak{h}$ with respect to this system (see [5, Part I]). Then every such function can be written as $m_1\Lambda_1 + \cdots + m_l\Lambda_l$ where m_i are all nonnegative integers. Let $H_1, \cdots, H_l$ be a base for $\mathfrak{h}$. Extend this to a base $X_1, \cdots, X_n$ ($n \geqq l$) for $\mathfrak{l}$ so that $X_i = H_i$, $1 \leqq i \leqq l$. Put g_{ij} = sp (ad X_i ad X_j), $1 \leqq i, j \leqq n$, where $X \rightarrow$ad X is the adjoint representation of $\mathfrak{l}$. Since $\mathfrak{l}$ is semisimple the matrix $(g_{ij})_{1 \leqq i,j \leqq n}$ is nonsingular. Let $(g^{ij})_{1 \leqq i,j \leqq n}$ denote its inverse. Let $\mathfrak{U}$ be the enveloping algebra of $\mathfrak{l}$. Put $\omega = \sum_{1 \leqq i,j \leqq n} g^{ij}X_iX_j \in \mathfrak{U}$. ω is called the Casimir operator of $\mathfrak{l}$ and it is well known that ω lies in the center of $\mathfrak{U}$. For any dominant integral function Λ on $\mathfrak{h}$ put

$$|\Lambda|^2 = \sum_{1 \leqq i,j \leqq l} g^{ij}\Lambda(H_i)\Lambda(H_j).$$

Then it is known (see for example [4]) that $|\Lambda|^2$ is a positive real number unless $\Lambda = 0$. Now let σ be an irreducible finite-dimensional representation of $\mathfrak{U}$ and let $\mathfrak{D}$ be the class of σ. We denote by $\Lambda_{\mathfrak{D}}$ the highest weight of σ and by $\omega_{\mathfrak{D}}$ the number such that $\sigma(\omega) = \omega_{\mathfrak{D}}\sigma(1)$. Then it follows from Lemma 6 of [4] that $\omega_{\mathfrak{D}}$ is real, $\omega_{\mathfrak{D}} \geqq |\Lambda_{\mathfrak{D}}|^2$, and there exists a real number κ such that $\kappa d(\mathfrak{D})^2 \geqq \omega_{\mathfrak{D}}$ for every irreducible class $\mathfrak{D}$. Hence

$$d(\mathfrak{D})^{-1} \leqq \kappa^{1/2} |\Lambda_{\mathfrak{D}}|^{-1}.$$

Now the base $H_1, \cdots, H_l$ can be so chosen that every root of $\mathfrak{l}$ takes real values at $H_1, \cdots, H_l$. For such a base the quadratic form $\sum_{i,j=1}^{l} g^{ij}x_i \cdot x_j$ ($x_i \in R$) is real and positive definite. We can therefore select $H_1, \cdots, H_l$ in such a way that this form reduces to $x_1^2 + \cdots + x_l^2$. For any dominant integral function Λ let e_Λ denote the vector in the l-dimensional real Euclidean space with the components $\Lambda(H_i)$. Then the set of all points e_Λ form one "octant" of a lattice whose generators are $e_i = e_{\Lambda_i}$, $1 \leqq i \leqq l$. Now

$$\sum_{\mathfrak{D}}' d(\mathfrak{D})^{-(l+1)} \leqq \kappa^{(l+1)/2} \sum_{\mathfrak{D}}' |\Lambda_{\mathfrak{D}}|^{-(l+1)}$$

where $\sum_{\mathfrak{D}}'$ denotes the sum over all irreducible classes $\mathfrak{D}$ except the one corresponding to the zero representation of degree 1. Since each class is completely determined by its highest weight, it follows that

$$\sum_{\mathfrak{D}}' |\Lambda_{\mathfrak{D}}|^{-(l+1)} \leqq \sum_{(m)\geqq 0}' |m_1 e_1 + \cdots + m_l e_l|^{-(l+1)}$$

where $|e|$ is the Euclidean length of the vector e and $\sum_{(m)\geqq 0}'$ denotes summation over all sets of nonnegative integers $(m_1, \cdots, m_l)$ such that $m_1 + \cdots + m_l > 0$. Since the series on the right is well known to be convergent, the lemma follows.

4. A result on convergence. We use the terminology of [6, §9]. Let π be a permissible representation of G on a Banach space $\mathfrak{H}$ and $E_{\mathfrak{D}}$ the canonical projection of $\mathfrak{H}$ on the space $\mathfrak{H}_{\mathfrak{D}}$ consisting of all elements in $\mathfrak{H}$ which transform under $\pi(K)$ according to $\mathfrak{D}$ $(\mathfrak{D}\in\Omega)$. We shall now prove the following lemma(8).

LEMMA 3. *There exists an element $z\in\mathfrak{X}$ such that*

$$\sum_{\mathfrak{D}\in\Omega} |E_{\mathfrak{D}}\psi| \leqq |\pi(z)\psi|$$

for any differentiable element ψ in $\mathfrak{H}$. Moreover the series

$$\sum_{\mathfrak{D}\in p} E_{\mathfrak{D}}\psi$$

converges to ψ.

Let $u\rightarrow u^*$ $(u\in K)$ denote the natural mapping of K on $K^*=K/D\cap Z$. For any $u\in K$ we denote by $\Gamma(u)$ the unique element in $\mathfrak{c}_0$ such that $u\exp(-\Gamma(u))\in K'$. Choose a base $\Gamma_1, \cdots, \Gamma_r$ for $\mathfrak{c}_0$ over R such that $\exp\Gamma_i$, $1\leqq i\leqq r$, is a set of generators for $D\cap Z$. Let μ be a linear function on $\mathfrak{c}$ such that $\pi(\exp\Gamma_i)=e^{\mu(\Gamma_i)}\pi(1)$, $1\leqq i\leqq r$. Let Ω_π be the set of all classes in Ω which occur in π. Then it is clear that if $\mathfrak{D}\in\Omega_\pi$ and σ is any representation in $\mathfrak{D}$, we must have

$$\sigma(\Gamma_i) = (2\pi(-1)^{1/2}n_i + \mu(\Gamma_i))\sigma(1)$$

where n_i, $1\leqq i\leqq r$, are all integers. Define a linear function $n_{\mathfrak{D}}$ on $\mathfrak{c}$ by setting $n_{\mathfrak{D}}(\Gamma_i)=n_i$, $1\leqq i\leqq r$, and put $|n_{\mathfrak{D}}|=(1+n_1^2+\cdots+n_r^2)^{1/2}$. Then if $w=1-(1/4\pi^2)\sum_{i=1}^r(\Gamma_i-\mu(\Gamma_i))^2\in\mathfrak{X}$, $\sigma(w)=|n_{\mathfrak{D}}|^2\sigma(1)$. We note that w lies in the center of $\mathfrak{X}$. Let $\mathfrak{X}'$ be the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{k}')$. Since $\mathfrak{k}'$ is semisimple we can find (see Lemma 4 of [7]) an element z_0 in the center of $\mathfrak{X}'$ such that $\sigma(z_0)=d_\sigma^2\sigma(1)$ for any simple representation σ of $\mathfrak{X}$ of degree d_σ.

Put $\pi^*(u^*)=e^{-\mu(\Gamma(u))}\pi(u)$ $(u\in K)$. Then π^* is a representation of K^* on

(8) Cf. Lemma 31 of [6] which was stated without proof.

$\mathfrak{H}$ and if $\mathfrak{D}\in\Omega_\pi$

$$E_{\mathfrak{D}} = d(\mathfrak{D})\int_{K^*}\text{conj }(\xi_{\mathfrak{D}}(u^*))\pi^*(u^*)du^*\text{(}^9\text{)}$$

where $\xi_{\mathfrak{D}}$ is the character (on K^*) of the class according to which every element in $\mathfrak{H}_{\mathfrak{D}}$ transforms under $\pi^*(K^*)$. Let M be an upper bound for $|\pi^*(u^*)|$ on the compact set K^*. Then it is clear that

$$|E_{\mathfrak{D}}| \leqq d(\mathfrak{D})^2M.$$

Let q and s be two integers $\geqq 0$. Then

$$|E_{\mathfrak{D}}\pi(z_0^{q+1}w^s)\psi| \leqq Md(\mathfrak{D})^2|\pi(z_0^{q+1}w^s)\psi|.$$

But if $X\in\mathfrak{k}_0$,

$$\lim_{t\to 0}\frac{1}{t}(\pi(\exp tX)-1)E_{\mathfrak{D}}\psi = \lim_{t\to 0}E_{\mathfrak{D}}\frac{1}{t}(\pi(\exp tX)-1)\psi = E_{\mathfrak{D}}\pi(X)\psi$$

since $E_{\mathfrak{D}}$ commutes with $\pi(u)$ $(u\in K)$. Hence it follows that $E_{\mathfrak{D}}\psi$ is differentiable under $\pi(K)$ and $\pi(x)E_{\mathfrak{D}}\psi = E_{\mathfrak{D}}\pi(x)\psi$ $(x\in\mathfrak{X})$. Therefore

$$E_{\mathfrak{D}}\pi(z_0^{q+1}w^s)\psi = \pi(z_0^{q+1}w^s)E_{\mathfrak{D}}\psi = d(\mathfrak{D})^{2q+2}|n_{\mathfrak{D}}|^{2s}E_{\mathfrak{D}}\psi$$

since $E_{\mathfrak{D}}\psi$ transforms under $\pi(K)$ according to $\mathfrak{D}$. Hence

$$d(\mathfrak{D})^{2q}|n_{\mathfrak{D}}|^{2s}|E_{\mathfrak{D}}\psi| \leqq M|\pi(z_0^{q+1}w^s)\psi| \qquad (\mathfrak{D}\in\Omega_\pi),$$

and therefore

$$\sum_{\mathfrak{D}\in\Omega_\pi}|E_{\mathfrak{D}}\psi| \leqq \left(\sum_{\mathfrak{D}\in\Omega_\pi}d(\mathfrak{D})^{-2q}|n_{\mathfrak{D}}|^{-2s}\right)M|\pi(z_0^{q+1}w^s)\psi|.$$

For any $\mathfrak{D}\in\Omega_\pi$ let $\mathfrak{D}'$ denote the class of representations of $\mathfrak{k}'$ defined as follows. If $\sigma\in\mathfrak{D}$, $\mathfrak{D}'$ is the class of the restriction of σ on $\mathfrak{k}'$. Clearly $\mathfrak{D}'$ is irreducible and $d(\mathfrak{D}')=d(\mathfrak{D})$. Moreover $\mathfrak{D}$ is completely determined by $\mathfrak{D}'$ and $n_{\mathfrak{D}}$. Hence

$$\sum_{\mathfrak{D}\in\Omega_\pi}d(\mathfrak{D})^{-2q}|n_{\mathfrak{D}}|^{-2s} \leqq \sum_{\mathfrak{D}'}d(\mathfrak{D}')^{-2q}\sum_{n_1,\cdots,n_r}(1+n_1^2+\cdots+n_r^2)^{-s}$$

where $\mathfrak{D}'$ runs over all irreducible classes of finite-dimensional representations of $\mathfrak{k}'$. But if $2q$ exceeds the rank of $\mathfrak{k}'$ it follows from Lemma 2 that $\sum'_{\mathfrak{D}}d(\mathfrak{D}')^{-2q}<\infty$. Similarly if $2s>r$

$$\sum_{(n)}(1+n_1^2+\cdots+n_r^2)^{-s} \leqq \sum_{(n)}(1+n_1^2+\cdots+n_r^2)^{-(r+1)/2} < \infty.$$

(9) Conj (x) means conjugate of x.

Therefore if we choose q and s sufficiently large and put

$$z = N z_0^{q+1} w^s$$

where

$$N = M \sum_{\mathfrak{D} \in \Omega_\pi} d(\mathfrak{D})^{-2q} | n_{\mathfrak{D}} |^{-2s},$$

$$\sum_{\mathfrak{D} \in \Omega} | E_{\mathfrak{D}} \psi | \leqq | \pi(z) \psi |.$$

This proves the first assertion of the lemma. Now we come to the second part. Since $\sum_{\mathfrak{D} \in \Omega} |E_{\mathfrak{D}}\psi| < \infty$ the series $\sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}}\psi$ is convergent. Let ϕ denote its sum. We have to show that $\phi = \psi$. Put $\psi' = \psi - \phi$. Since $E_{\mathfrak{D}}\phi = E_{\mathfrak{D}}\psi$, $E_{\mathfrak{D}}\psi' = 0$. From this we shall deduce that $\psi' = 0$.

Suppose $\psi' \neq 0$. Then given any real $\epsilon > 0$ choose a continuous real non-negative function f on K^* such that $f(u^*) = 0$ if $|\pi^*(u^*)\psi' - \psi'| > \epsilon |\psi'|$ $(u^* \in K^*)$ and $\int_{K^*} f(u^*) du^* = 1$. Moreover choose a finite linear combination ω of the matrix coefficients of finite-dimensional simple representations of K^* such that $|f(u^*) - \omega(u^*)| \leqq \epsilon$ $(u^* \in K^*)$. Then if

$$\psi'' = \int \omega(u^*) \pi^*(u^*) \psi' du^*,$$

$\psi'' \in \sum_{\mathfrak{D} \in \Omega} \mathfrak{H}_{\mathfrak{D}}$ and therefore $\psi'' = \sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}} \psi''$. But

$$E_{\mathfrak{D}}\psi'' = \int \omega(u^*) \pi^*(u^*) E_{\mathfrak{D}} \psi' du^* = 0$$

since $E_{\mathfrak{D}}\psi' = 0$. Hence $\psi'' = 0$. On the other hand

$$\begin{aligned} | \psi'' - \psi' | &\leqq \int | \omega(u^*) - f(u^*) | \, | \pi^*(u^*) \psi' | \, du^* \\ &\quad + \int f(u^*) \, | \pi^*(u^*) \psi' - \psi' | \, du^* \\ &\leqq M\epsilon | \psi' | + \epsilon | \psi' | \end{aligned}$$

where $M = \sup_{u^* \in K^*} |\pi^*(u^*)|$. Therefore if ϵ is sufficiently small

$$| \psi' | = | \psi'' - \psi' | \leqq | \psi' | /2$$

which contradicts our assumption that $\psi' \neq 0$. Therefore $\psi' = 0$ and so $\sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}} \psi$ converges to ψ.

5. **Characters.** Let $C_c^\infty(G)$ denote the class of all complex-valued functions on G which are indefinitely differentiable everywhere and which vanish outside a compact set. Let π be a quasi-simple irreducible representation of G on a Hilbert space $\mathfrak{H}$. For any $f \in C_c^\infty(G)$ consider the operator

$$T_f = \int f(x)\pi(x)dx$$

where dx is the Haar measure on G. We intend to show that T_f is of the trace class.

Let $\mathfrak{O}'$ be the Banach space of all bounded linear operators A on $\mathfrak{H}$ with the usual norm $|A| = \sup_{|\psi|\leq 1} |A\psi|$ $(\psi \in \mathfrak{H})$. Let $\mathfrak{O}_0$ be the subspace of $\mathfrak{O}'$ consisting of all operators of the form T_f $(f \in C_c^\infty(G))$. We denote by $\mathfrak{O}$ the closure of $\mathfrak{O}_0$ in $\mathfrak{O}'$. Now if $y \in G$,

$$\pi(y)T_f = \int f(y^{-1}x)\pi(x)dx, \qquad T_f\pi(y^{-1}) = \int f(xy)\pi(x)dx.$$

Hence it follows that if $A \in \mathfrak{O}$ then $\pi(y)A$ and $A\pi(y^{-1})$ are also in $\mathfrak{O}$. We now define two representations l and r of G on $\mathfrak{O}$ as follows:

$$l(x)A = \pi(x)A, \qquad r(x)A = A\pi(x^{-1}) \qquad (x \in G, A \in \mathfrak{O}).$$

In order to verify the conditions for continuity it is sufficient to prove that $\lim_{x\to 1, y \to 1} |\pi(x)A\pi(y^{-1}) - A| = 0$ $(A \in \mathfrak{O})$. This is done as follows. Given $\epsilon > 0$, choose $f \in C_c^\infty(G)$ such that $|A - T_f| \leq \epsilon$. Let $U = U^{-1}$ be a compact neighbourhood of 1 in G and M an upper bound for $|\pi(z)|$ for $z \in U$. Then

$$|\pi(x)A\pi(y^{-1}) - \pi(x)T_f\pi(y^{-1})| \leq M^2\epsilon \qquad (x, y \in U)$$

and therefore

$$|\pi(x)A\pi(y^{-1}) - A| \leq (M^2 + 1)\epsilon + |\pi(x)T_f\pi(y^{-1}) - T_f|$$

$$\leq (M^2 + 1)\epsilon + \int |f(x^{-1}zy) - f(z)| \, |\pi(z)| dz.$$

Let C be a compact set outside which f is zero. We can choose a neighbourhood V of 1 in G $(V \subset U)$ such that $|f(x^{-1}zy) - f(z)| \leq \epsilon$ if $x, y \in V$. Let F be a real nonnegative continuous function on G which is equal to 1 on C and which vanishes outside some compact set. Then if $N_0 = \sup_{z \in UCU} |\pi(z)|$,

$$|\pi(x)A\pi(y^{-1}) - A| \leq (M^2 + 1)\epsilon + N_0\epsilon \int F(z)dz$$

provided $x, y \in V$. This proves that $\lim_{x \to 1, y \to 1} |\pi(x)A\pi(y^{-1}) - A| = 0$.

Since $l(x)T_f = \int f(x^{-1}z)\pi(z)dz$, it follows easily that T_f is differentiable under l. Similarly we show that it is differentiable under r. It is clear that the representations l and r are permissible. For any $\mathfrak{D} \in \Omega$ let $E_\mathfrak{D}$, $P_\mathfrak{D}$, and $Q_\mathfrak{D}$ denote the canonical projections (see §9 of [6]) corresponding to $\mathfrak{D}$ under π, l, and r respectively. Then it is clear $P_\mathfrak{D}A = E_\mathfrak{D}A$; $Q_\mathfrak{D}A = AE_{\mathfrak{D}'}$ $(\mathfrak{D} \in \Omega)$ where $\mathfrak{D}'$ is the class contragredient to $\mathfrak{D}$. Let λ and ρ denote the left and right regular representations of G. Then every element in $C_c^\infty(G)$ is differentiable under both λ and ρ and $C_c^\infty(G)$ is invariant under $\lambda(\mathfrak{B})$ and $\rho(\mathfrak{B})$. Moreover since $l(x)r(y)T_f = T_{\lambda(x)\rho(y)f}$ $(x, y \in G)$ it follows easily that $l(a)r(b)T_f = T_{\lambda(a)\rho(b)f}$ $(a, b \in \mathfrak{B})$.

Now define a representation ϕ of the group $G \times G$ on $\mathfrak{O}$ as follows. $\phi(x, y)A = l(x)r(y)A = \pi(x)A\pi(y^{-1})$ $(x, y \in G)$. Then ϕ is a permissible representation of the semisimple group $G \times G$ and any element of $\mathfrak{O}_0$ is differentiable under ϕ. Moreover the canonical projections for the representation ϕ (with respect to the subgroup $K \times K$) are exactly the operators $P_{\mathfrak{D}_1}Q_{\mathfrak{D}_2}$ $(\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega)$. Let z_0 be the element of $\mathfrak{X}$ which was introduced in the proof of Lemma 3. Then if we apply Lemma 3 to the representation ϕ and the differentiable element $T_{\lambda(z_0)\rho(z_0)f}$ we find that

$$\sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} \left| P_{\mathfrak{D}_1}Q_{\mathfrak{D}_2}T_{\lambda(z_0)\rho(z_0)f} \right| < \infty.$$

But

$$P_{\mathfrak{D}_1}Q_{\mathfrak{D}_2}T_{\lambda(z_0)\rho(z_0)f} = P_{\mathfrak{D}_1}Q_{\mathfrak{D}_2}l(z_0)r(z_0)T_f = d(\mathfrak{D}_1)^2 d(\mathfrak{D}_2)^2 P_{\mathfrak{D}_1}Q_{\mathfrak{D}_2}T_f.$$

Hence

$$\sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} d(\mathfrak{D}_1)^2 d(\mathfrak{D}_2)^2 \left| E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2} \right| < \infty.$$

Now let us first suppose that the subspaces $\mathfrak{H}_{\mathfrak{D}} = E_{\mathfrak{D}}\mathfrak{H}$ $(\mathfrak{D} \in \Omega)$ are mutually orthogonal. Choose an orthonormal base for each $\mathfrak{H}_{\mathfrak{D}}$. All these put together form an orthonormal base $\{\psi_\alpha\}_{\alpha \in J}$ for $\mathfrak{H}$. In accordance with Theorem 4 we choose an integer N such that $\dim \mathfrak{H}_{\mathfrak{D}} \leqq Nd(\mathfrak{D})^2$ $(\mathfrak{D} \in \Omega)$. Then

$$\sum_{\alpha, \beta \in J} \left| (\psi_\alpha, T_f\psi_\beta) \right| = \sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} \sum_{\alpha \in J_{\mathfrak{D}_1}} \sum_{\beta \in J_{\mathfrak{D}_2}} \left| (\psi_\alpha, T_f\psi_\beta) \right|$$

where $J_{\mathfrak{D}}$ is the subset of J such that $\{\psi_\alpha\}_{\alpha \in J_{\mathfrak{D}}}$ is a base for $\mathfrak{H}_{\mathfrak{D}}$. But it is clear that

$$\sum_{\alpha \in J_{\mathfrak{D}_1}} \sum_{\beta \in J_{\mathfrak{D}_2}} \left| (\psi_\alpha, T_f\psi_\beta) \right| \leqq \dim \mathfrak{H}_{\mathfrak{D}_1} \dim \mathfrak{H}_{\mathfrak{D}_2} \left| E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2} \right|$$

$$\leqq N^2 d(\mathfrak{D}_1)^2 d(\mathfrak{D}_2)^2 \left| E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2} \right|.$$

Hence

$$\sum_{\alpha, \beta \in J} \left| \psi_\alpha, T_f\psi_\beta \right| \leqq N^2 \sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} d(\mathfrak{D}_1)^2 d(\mathfrak{D}_2)^2 \left| E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2} \right| < \infty$$

and therefore, from Lemma 1, T_f is of the trace class.

Now we discard the assumption about the mutual orthogonality of the spaces $\mathfrak{H}_{\mathfrak{D}}$. Let $x \rightarrow x^*$ denote the natural mapping of G on $G^* = G/D \cap Z$ Define a representation π^* of K^* on $\mathfrak{H}$ as in the proof of Lemma 3. Since K^* is a compact group, π^* is equivalent to a unitary representation. Hence there exists a regular operator B on $\mathfrak{H}$ such that the representation π'^*: $u \rightarrow B\pi^*(u^*)B^{-1}$ $(u^* \in K^*)$ is unitary. Now put $\pi'(x) = B\pi(x)B^{-1}$. Then π' is a representation of G on $\mathfrak{H}$. Let $\mathfrak{H}'_{\mathfrak{D}}$ be the subspace of $\mathfrak{H}$ consisting of all elements which transform under $\pi'(K)$ according to $\mathfrak{D}$ ($\mathfrak{D}\in\Omega$). Then since π'^* is unitary the spaces $\mathfrak{H}'_{\mathfrak{D}}$ are mutually orthogonal. Therefore the above proof is applicable to

$$T_f' = \int f(x)\pi'(x)dx = BT_fB^{-1}.$$

Hence T_f' fulfills the condition of Lemma 1 and therefore $T_f=B^{-1}T_f'B$ is of the trace class. Moreover if P and Q are two bounded operators on $\mathfrak{H}$, then PB^{-1} and BQ are also bounded and $PT_fQ=(PB^{-1})T_f'(BQ)$. Therefore, from Lemma 1, PT_fQ, T_fQP, QPT_f are all of the trace class and their traces are equal. Therefore in particular sp $(AT_fA^{-1})=$ sp T_f if A is a regular operator.

Put $T_\pi(f)=\mathrm{sp}(T_f)$ for any $f\in C_c^\infty(G)$. Then T_π is a linear function on the vector space $C_c^\infty(G)$. Furthermore if a is a fixed element in G and g is the function $g(x)=f(axa^{-1})$ $(x\in G)$ then

$$T_g = \pi(a^{-1})T_f\pi(a)$$

and therefore $T_\pi(g)=T_\pi(f)$. Hence we may say that T_π is invariant under the inner automorphisms of G. We prove similarly that if π and π' are equivalent representations, $T_\pi=T_{\pi'}$.

Our next object is to show that T_π is actually a distribution in the sense of L. Schwartz [9]. By going over to an equivalent representation, if necessary, we may assume that the spaces $\mathfrak{H}_{\mathfrak{D}}$ ($\mathfrak{D}\in\Omega$) are mutually orthogonal. Then it is clear that

$$\mathrm{sp}\ T_f = \sum_{\mathfrak{D}\in\Omega} \mathrm{sp}\ (E_{\mathfrak{D}}T_fE_{\mathfrak{D}})$$

and

$$\mathrm{sp}\ (E_{\mathfrak{D}}T_fE_{\mathfrak{D}}) = \sum_{i=1}^{d} (\psi_i, E_{\mathfrak{D}}T_f\psi_i)$$

where $(\psi_1, \cdots, \psi_d)$ is an orthonormal base of $\mathfrak{H}_{\mathfrak{D}}$. Therefore

$$|\mathrm{sp}\ (E_{\mathfrak{D}}T_fE_{\mathfrak{D}})| \leqq \dim \mathfrak{H}_{\mathfrak{D}} |E_{\mathfrak{D}}T_f| \leqq Nd(\mathfrak{D})^2|E_{\mathfrak{D}}T_f|$$

and

$$|T_\pi(f)| \leqq N\sum_{\mathfrak{D}\in\Omega} d(\mathfrak{D})^2|E_{\mathfrak{D}}T_f| = N\sum_{\mathfrak{D}\in\Omega} |E_{\mathfrak{D}}T_{\lambda(z_0)f}|$$

where z_0 has the same meaning as above. Now by applying Lemma 3 to the representation l of G on $\mathfrak{L}$ and the differentiable element $T_{\lambda(z_0)f}$ we conclude that

$$\sum_{\mathfrak{D}\in\Omega} |E_{\mathfrak{D}}T_{\lambda(z_0)f}| \leqq |l(z)T_{\lambda(z_0)f}| = |T_{\lambda(zz_0)f}|$$

where z is an element of $\mathfrak{X}$ (which does not depend on f). Hence

$$|T_\pi(f)| \leqq N |T_{\lambda(zz_0)f}|.$$

Now suppose C is a compact set in G and f_n is a sequence of functions in $C_c^\infty(G)$ such that f_n vanishes outside C and for any $b \in \mathfrak{B}$, $\lambda(b)f_n \to 0$ uniformly on C. Then

$$|T_{\lambda(b)f_n}| \leqq \int |(\lambda(b)f_n)(x)| |\pi(x)| dx \to 0$$

and therefore in particular $|T_\pi(f_n)| \leqq N |T_{\lambda(zz_0)f_n}| \to 0$. This proves that T_π is a distribution.

6. **Operators of the Hilbert-Schmidt class.** Let B be a bounded operator on the Hilbert space $\mathfrak{H}$ and let B^* be the adjoint of B. We say that B is of the Hilbert-Schmidt (H.S.) class if B^*B has a trace. Let $\{\psi_\alpha\}_{\alpha \in J}$ be an orthonormal base for $\mathfrak{H}$. Then it is well known that $\|B\|^2 = \sum_{\alpha \in J} |B\psi_\alpha|^2$ is independent of the choice of this base and B is of the H.S. class if and only if $\|B\| < \infty$. Moreover sp $BB^* = \|B\|^2 = \|B^*\|^2$ if $\|B\| < \infty$ and $\|A_1BA_2\| \leqq |A_1| \|B\| |A_2|$ for any two bounded operators A_1, A_2.

Let π be a quasi-simple irreducible representation of G on $\mathfrak{H}$. Let f be a complex-valued measurable function on G which vanishes outside a compact set and such that $\int |f(x)|^2 dx < \infty$. It follows from the Schwartz inequality that $\int |f(x)| dx < \infty$ and therefore the operator $\int f(x)\pi(x)dx$ is a well-defined bounded operator. We intend to prove that this operator is of the H.S. class. Let S be a regular operator on $\mathfrak{H}$. Put $\pi'(x) = S\pi(x)S^{-1}$ $(x \in G)$. Then

$$\left\| \int f(x)\pi(x)dx \right\| = \left\| S^{-1} \int f(x)\pi'(x)dxS \right\| \leqq |S^{-1}| \left\| \int f(x)\pi(x)dx \right\| |S|.$$

Therefore it is enough to show that the corresponding operator for an equivalent representation is of the H.S. class.

Let $x \to x^*$ denote the natural mapping of G on $G^* = G/D \cap Z$. For any $x \in G$ we define $\Gamma(x)$ to be the unique element in $\mathfrak{c}_0$ such that $x = u(\exp \Gamma(x))s$ where $u \in K'$ and s lies in the solvable subgroup of G corresponding to the subalgebra $\mathfrak{g}_0 \cap (\mathfrak{h}_\mathfrak{p} + \mathfrak{n})$ of $\mathfrak{g}_0$ (see [6, §9]). Then if μ is the linear function on $\mathfrak{c}$ which was introduced in the proof of Lemma 3, it is clear that $\pi(x)e^{-\mu(\Gamma(x))}$ depends only on x^*. Put $\pi^*(x^*) = \pi(x)e^{-\mu(\Gamma(x))}$. Then we verify immediately that $\pi^*(u^*x^*) = \pi^*(u^*)\pi^*(x^*)$ $(u^* \in K^*, x^* \in G^*)$ and therefore $u^* \to \pi^*(u^*)$ is a representation of K^*. In view of the preceding remarks we may assume without loss of generality that this representation is unitary.

Let $x^* \in G^*$ and $y \in G$. We say that y lies above x^* and write $y > x^*$ if $(y)^* = x^*$. Put

$$f^*(x^*) = \sum_{x > x^*} e^{\mu(\Gamma(x))} f(x) \qquad (x^* \in G^*).$$

Let A be a compact set outside which f is zero. Since $D\cap Z$ is discrete, $(D\cap Z)\cap A^{-1}A$ is a finite set. Let N_0 be the number of elements in it. Then it is clear that not more than N_0 distinct elements in A can lie above the same element in G^*. Hence at most N_0 terms in the above sum are different from zero and therefore the function f^* is well-defined. Moreover if A^* is the image of A in G^*, f^* is zero outside A^*. Now let $x^*\in A^*$. Then

$$|f^*(x^*)| = \left|\sum_{x>x^*} e^{\mu(\Gamma(x))}f(x)\right| \leqq \left(\sum_{x>x^*} |f(x)|^2\right)^{1/2}\left(\sum_{y>x^*} |e^{\mu(\Gamma(y))}|^2\right)^{1/2}$$

$$\leqq M_0N_0^{1/2}\left(\sum_{x>x^*} |f(x)|^2\right)^{1/2}$$

where $M_0=\sup_{y\in A}\left|e^{\mu(\Gamma(y))}\right|$. Hence if the Haar measure dx^* on G^* is suitably normalised it follows that

$$\int f(x)\pi(x)dx = \int f^*(x^*)\pi^*(x^*)dx^*$$

and

$$\int |f^*(x^*)|^2dx^* \leqq M_0^2N\int |f(x)|^2dx < \infty.$$

Let B^* be a compact neighbourhood of A^*. Choose a real-valued nonnegative function F on G^* such that $F=1$ on K^*B^* and $F=0$ outside some compact set. Let g be any continuous function on G^* which vanishes outside B^*. Consider the operator

$$\int g(x^*)\pi(x^*)dx^* = \int du^*\int g(u^*x^*)\pi^*(u^*x^*)dx^*.$$

(Here du^* is the normalised Haar measure on K^* so that $\int du^*=1$.) Then

$$\left\|\int g(x^*)\pi^*(x^*)dx^*\right\| \leqq \int dx^*\left\|\int g(u^*x^*)\pi^*(u^*x^*)du^*\right\|.$$

But

$$\left\|\int g(u^*x^*)\pi^*(u^*x^*)du^*\right\| \leqq \left\|\int g(u^*x^*)\pi^*(u^*)du^*\right\| |\pi^*(x)|$$

and from Theorem 4 we can find an integer N such that dim $\mathfrak{H}_{\mathfrak{D}}\leqq Nd(\mathfrak{D})^2$ for any $\mathfrak{D}\in\Omega$. Let Ω^* be the set of all classes of irreducible finite-dimensional representations of K^*. Then no $\mathfrak{D}^*\in\Omega^*$ occurs more than $Nd(\mathfrak{D}^*)$ times in the reduction of $\pi^*(K^*)$. Since every $\mathfrak{D}^*\in\Omega^*$ occurs exactly $d(\mathfrak{D}^*)$ times in the left regular representation λ of K^* (on the Hilbert space $L_2(K^*)$ of all square integrable functions on K^*) and since the representation $u^*\to\pi^*(u^*)$ $(u^*\in K^*)$ is unitary, we may conclude that

$$\left\|\int g(u^*x^*)\pi^*(u^*)du^*\right\|^2 \leqq N\left\|\int g(u^*x^*)\lambda(u^*)du^*\right\|^2.$$

But from the Peter-Weyl theorem we know that

$$\left\|\int g(u^*x^*)\lambda(u^*)du^*\right\|^2 = \int |g(u^*x^*)|^2 du^*.$$

Hence

$$\left\|\int g(u^*x^*)\pi^*(u^*)du^*\right\| \leqq N^{1/2}\left(\int |g(u^*x^*)|^2 du^*\right)^{1/2}.$$

Now it is easy to see that $|\pi^*(x^*)|$ is bounded on the compact set K^*B^*. Let $M=\sup_{x^*\in K^*B^*}|\pi^*(x^*)|$. Then

$$\left\|\int g(u^*x^*)\pi^*(u^*x^*)du^*\right\| \leqq MN^{1/2}\left(\int |g(u^*x^*)|^2 du^*\right)^{1/2}$$

and therefore

$$\begin{aligned}\left\|\int g(x^*)\pi^*(x^*)dx^*\right\| &\leqq MN^{1/2}\int dx^*\left(\int |g(u^*x^*)|^2 du^*\right)^{1/2}\\ &= MN^{1/2}\int F(x^*)dx^*\left(\int |g(u^*x^*)|^2 du^*\right)^{1/2}\\ &\leqq M_1\left(\int |g(x^*)|^2 dx^*\right)^{1/2}\end{aligned}$$

where $M_1=MN^{1/2}(\int|F(x^*)|^2dx^*)^{1/2}$. Now choose a sequence g_n of continuous functions on G^* which vanish outside B^* and such that $\int|f^*(x^*)-g_n(x^*)|^2dx^*\to 0$. Then since

$$\int |f^*(x^*) - g_n(x^*)| dx^* \leqq \left(\int |F(x^*)|^2dx^*\right)^{1/2}\left(\int |f^*(x^*) - g_n(x^*)|^2dx^*\right)^{1/2}$$

it follows that

$$\left|\int (f^*(x^*) - g_n(x^*))\pi^*(x^*)dx^*\right| \to 0.$$

Moreover we have seen above that

$$\left\| \int (g_m(x^*) - g_n(x^*))\pi^*(x^*)dx^* \right\| \leqq M_1 \left(\int | g_m(x^*) - g_n(x^*) |^2 dx^* \right)^{1/2}$$

and therefore the sequence of operators $T_n = \int g_n(x^*)\pi^*(x^*)dx^*$ is a Cauchy sequence with respect to the Hilbert-Schmidt norm $\| \quad \|$. Since the space of operators of the H.S. class is complete with respect to this norm, there exists an operator T of this class such that $\|T_n - T\| \to 0$. But $|T_n - T| \leqq \|T_n - T\|$. Hence $|T_n - T| \to 0$. However we have seen already that

$$\left| T_n - \int f^*(x^*)\pi^*(x^*)dx^* \right| \to 0$$

and therefore $T = \int f^*(x^*)\pi^*(x^*)dx^*$. This proves that $T = \int f(x)\pi(x)dx$ is of the H.S. class. Thus we have the following theorem(10).

THEOREM 5. *Let π be a quasi-simple irreducible representation of G on a Hilbert space $\mathfrak{H}$ and let f be a measurable and square integrable function on G which vanishes outside a compact set. Then the operator $\int f(x)\pi(x)dx$ is of the Hilbert-Schmidt class.*

7. **Linear independence of characters.** Let T_π be the character of a quasi-simple irreducible representation π of G on a Hilbert space $\mathfrak{H}$. Let $E_{\mathfrak{D}}$ denote the canonical projection of $\mathfrak{H}$ on $\mathfrak{H}_{\mathfrak{D}}$ ($\mathfrak{D} \in \Omega$). Then it follows easily from its definition (see §5) that

$$T_\pi(f) = \operatorname{sp}\left(\int f(x)\pi(x)dx \right) = \sum_{\mathfrak{D} \in \Omega} \operatorname{sp}\left(E_{\mathfrak{D}} \int f(x)\pi(x)dx \cdot E_{\mathfrak{D}} \right)$$

$$= \sum_{\mathfrak{D} \in \Omega} \int f(x)\phi_{\mathfrak{D}}^{\pi}(x)dx$$

in the notation of §1. Now if π_1, π_2 are two infinitesimally equivalent representations, we have seen in §1 that $\phi_{\mathfrak{D}}^{\pi_1} = \phi_{\mathfrak{D}}^{\pi_2}$ ($\mathfrak{D} \in \Omega$) and therefore $T_{\pi_1} = T_{\pi_2}$. Hence two infinitesimally equivalent quasi-simple irreducible representations (on Hilbert spaces) have the same character. Conversely we shall show that two such representations having the same character are infinitesimally equivalent(11).

THEOREM 6. *Let $\pi_1, \cdots, \pi_q$ be a finite set of quasi-simple irreducible representations of G on the Hilbert spaces $\mathfrak{H}_1, \cdots, \mathfrak{H}_q$ respectively. Suppose no two of them are infinitesimally equivalent. Then their characters $T_{\pi_1}, \cdots, T_{\pi_q}$ are linearly independent.*

For otherwise suppose $c_1 T_{\pi_1} + \cdots + c_q T_{\pi_q} = 0$ ($c_i \in C$) where, say, $c_1 \neq 0$.

(10) Cf. Theorem 4 of [8(c)].
(11) Cf. Theorem 3 of [8(c)].

Let η_i be the homomorphism of $D\cap Z$ into C such that $\pi_i(\gamma)=\eta_i(\gamma)\pi_i(1)$ $(\gamma\in D\cap Z)$. Then if $f\in C_c^\infty(G)$ and $\gamma\in D\cap Z$, the function f_γ: $x\to f(\gamma^{-1}x)$ $(x\in G)$ is also in $C_c^\infty(G)$ and it is obvious that $T_{\pi_i}(f_\gamma)=\eta_i(\gamma)T_{\pi_i}(f)$. Therefore $\sum_{i=1}^q c_iT_{\pi_i}(f_\gamma)=\sum_{i=1}^q c_i\eta_i(\gamma)T_{\pi_i}(f)=0$. This proves that

$$\sum_{i=1}^q c_i\eta_i(\gamma)T_{\pi_i} = 0$$

for all $\gamma\in D\cap Z$. Now if we recall that $D\cap Z$ is a free abelian group with r generators ($r=\dim_R \mathfrak{c}_0$) we can conclude that $c_1T_{\pi_1}+\cdots+c_sT_{\pi_s}=0$ assuming that $\eta_j=\eta_1$ $(1\leqq j\leqq s)$ and $\eta_j\neq\eta_i$ for $s<j\leqq q$.

Choose a base $\Gamma_1, \cdots, \Gamma_r$ for $\mathfrak{c}_0$ over R such that $\exp \Gamma_i$, $1\leqq i\leqq r$, is a set of generators for $D\cap Z$. Select a linear function μ on $\mathfrak{c}$ such that $\eta_1(\exp \Gamma_i)=e^{\mu(\Gamma_i)}$, $1\leqq i\leqq r$. Put $\pi_j^*(x^*)=e^{-\mu(\Gamma(x))}\pi_j(x)$ $(x\in G,\ 1\leqq j\leqq s)$ in the notation of §6. Let $\mathfrak{D}$ be a class in Ω which occurs in π_1. We denote by $E_{\mathfrak{D}}^i$ the canonical projection of $\mathfrak{H}_i$ on $\mathfrak{H}_{i,\mathfrak{D}}$. Then

$$E_{\mathfrak{D}}^i = d(\mathfrak{D})\int \text{conj }(\xi_{\mathfrak{D}^*}(u^*))\pi_i^*(u^*)du^*$$

where $\mathfrak{D}^*$ is the irreducible class according to which every element in $\mathfrak{H}_{1,\mathfrak{D}}$ transforms under $\pi_1^*(K^*)$ and $\xi_{\mathfrak{D}^*}$ is the character of $\mathfrak{D}^*$. Let K_0 be the set of all elements in K of the form $(\exp \Gamma)v$ where $\Gamma=t_1\Gamma_1+\cdots+t_r\Gamma_r$ $(t_j\in R,$ $|t_j|\leqq 1/2)$ and $v\in K'$. Then K_0 is compact and if we put

$$\xi_{\mathfrak{D}}(u) = e^{-\text{conj}(\mu(\Gamma(u)))}\xi_{\mathfrak{D}^*}(u^*) \qquad (u\in K),$$

we get

$$E_{\mathfrak{D}}^i = d(\mathfrak{D})\int_{K_0} \text{conj }(\xi_{\mathfrak{D}}(u))\pi_i(u)du$$

where the Haar measure du on K is so normalised that $\int_{K_0}du=1$.

Now we use the notation of Theorem 1. Put

$$\phi = c_1\phi_{\mathfrak{D}}^{\pi_1}+\cdots+c_s\phi_{\mathfrak{D}}^{\pi_1}.$$

It follows from Theorem 1 that $\phi\neq 0$. Since ϕ is continuous we can find a function $f\in C_c^\infty(G)$ such that $\int f(x)\phi(x)dx\neq 0$. Now

$$E_{\mathfrak{D}}^i\int f(x)\pi_i(x)dx = \int f_{\mathfrak{D}}(x)\pi_i(x)dx \qquad (1\leqq i\leqq s)$$

where

$$f_{\mathfrak{D}}(x) = d(\mathfrak{D})\int_{K_0} \text{conj }(\xi_{\mathfrak{D}}(u))f(u^{-1}x)du.$$

Since K_0 is compact it is clear that $f_{\mathfrak{D}}\in C_c^\infty(G)$. On the other hand

$$\operatorname{sp}\left(\int f_{\mathfrak{D}}(x)\pi_i(x)dx\right) = \operatorname{sp}\left(E^i_{\mathfrak{D}}\int f(x)\pi_i(x)dx\right)$$

$$= \operatorname{sp}\left(E^i_{\mathfrak{D}}\int f(x)\pi_i(x)dx\cdot E^i_{\mathfrak{D}}\right) = \int f(x)\phi^{\pi_i}_{\mathfrak{D}}(x)dx.$$

Therefore $c_1T_{\pi_1}(f_{\mathfrak{D}}) + \cdots + c_sT_{\pi_s}(f_{\mathfrak{D}}) = \int f(x)\phi(x)dx \neq 0$. This however implies that $c_1T_{\pi_i}+ \cdots +c_sT_{\pi_s}\neq 0$ and so we get a contradiction.

COROLLARY. *Two irreducible unitary representations are equivalent if and only if their characters are the same.*

First of all every irreducible unitary representation is quasi-simple (see for example Segal [10]). Moreover infinitesimal equivalence is the same as ordinary equivalence for two such representations (see Theorem 8 of [6]). Hence the corollary is an immediate consequence of the theorem.

8. **Complex semisimple groups.** Suppose the group G is complex. Then K is semisimple and there exists a 1-1 linear mapping i of $\mathfrak{k}_0$ onto $\mathfrak{p}_0$ such that

$$[X, i(Y)] = i([X, Y]), \qquad [i(X), i(Y)] = -[X, Y] \qquad (X, Y \in \mathfrak{k}_0).$$

Let $\mathfrak{h}_{\mathfrak{k}_0}$ be a maximal abelian subalgebra of $\mathfrak{k}_0$. Then $i(\mathfrak{h}_{\mathfrak{k}_0})$ is clearly a maximal abelian subspace of $\mathfrak{p}_0$. Hence we may take $\mathfrak{h}_{\mathfrak{p}_0}=i(\mathfrak{h}_{\mathfrak{k}_0})$. Then $\mathfrak{h}_{\mathfrak{k}_0}+\mathfrak{h}_{\mathfrak{p}_0}$ is a maximal abelian subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{h}_{\mathfrak{k}}$ and $\mathfrak{h}_{\mathfrak{p}}$ be the subspaces of $\mathfrak{g}$ spanned by $\mathfrak{h}_{\mathfrak{k}_0}$ and $\mathfrak{h}_{\mathfrak{p}_0}$ over C. Then $\mathfrak{h}=\mathfrak{h}_{\mathfrak{k}}+\mathfrak{h}_{\mathfrak{p}}$ is a Cartan subalgebra of $\mathfrak{g}$. We extend i to a mapping of $\mathfrak{k}$ into $\mathfrak{p}$ by linearity.

Let $\alpha_1, \cdots, \alpha_p$ be a maximal set of linearly independent roots of $\mathfrak{k}$ (with respect to $\mathfrak{h}_{\mathfrak{k}}$). We order all roots α of $\mathfrak{k}$ lexicographically with respect to this set (see [5, Part I]). For every root α let H_α be the element in $\mathfrak{h}_{\mathfrak{k}}$ such that $\operatorname{sp}(\operatorname{ad}' H \operatorname{ad}' H_\alpha)=\alpha(H)$ $(H\in\mathfrak{h}_{\mathfrak{k}})$ where $X\to\operatorname{ad}' X$ $(X\in\mathfrak{k})$ is the adjoint representation of $\mathfrak{k}$. We denote by W the Weyl group of $\mathfrak{k}$ and by 2σ the sum of all positive roots of $\mathfrak{k}$. Let λ be a linear function on $\mathfrak{h}_{\mathfrak{k}}$. We put $\lambda'=\lambda+\sigma$ and use the notation of [5, Part III]. We know (see [5, p. 70]) that the power series $\sum_{s\in W}\epsilon(s)e^{\lambda'(sH)}$ is divisible by $\prod_{\alpha>0}\lambda'(H_\alpha)\prod_{\alpha>0}\alpha(H)$ $(H\in\mathfrak{h}_{\mathfrak{k}})$ and therefore the quotient

$$\frac{\prod_{s\in W}\epsilon(s)e^{\lambda'(sH)}}{\prod_{\alpha>0}\lambda'(H_\alpha)\prod_{\alpha>0}\alpha(H)}$$

is an analytic function on $\mathfrak{h}_{\mathfrak{k}}$. Similarly

$$\frac{\prod_{\alpha>0}\alpha(H)}{\prod_{\alpha>0}(e^{\alpha(H)/2}-e^{-\alpha(H)/2})}$$

is a meromorphic function on $\mathfrak{h}_{\mathfrak{k}}$ all whose singularities lie on hyperplanes of the form $\alpha(H)=2\pi(-1)^{1/2}n$ where α is a root and n is some nonzero integer. Hence the function

$$\Phi^*(\lambda, H) = \prod_{\alpha>0} \sigma(H_\alpha) \frac{\sum_{s\in W} \epsilon(s)e^{\lambda'(sH)}}{\prod_{\alpha>0} \lambda'(H_\alpha) \prod_{\alpha>0} \alpha(H)} \frac{\prod_{\alpha>0} \alpha(H)}{\prod_{\alpha>0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

is a meromorphic function on $\mathfrak{h}_{\mathfrak{k}}$ and it is analytic everywhere on $(-1)^{1/2}\mathfrak{h}_{\mathfrak{k}_0}$ and also on a suitable neighbourhood of zero in $\mathfrak{h}_{\mathfrak{k}}$. We know from [5, Part III, p. 71] that if $H\in\mathfrak{h}_{\mathfrak{k}}$ and $|t|$ is sufficiently small ($t\in C$) then

$$\Phi^*(\lambda, tH) = \sum_{m\geq 0} \frac{t^m}{m!} \xi_\lambda(H^m)$$

where ξ_λ is the (infinitesimal) character of the algebra $\mathfrak{X}$ corresponding to the linear function λ on $\mathfrak{h}_{\mathfrak{k}}$.

Λ being any linear function on $\mathfrak{h}_{\mathfrak{p}}$, consider the integral $\int_K e^{\Lambda(H(x,u))}du$ which occurs in Theorem 2. (Notice that $K=K^*$ in our case and therefore $\int_K du=1$.) We shall now express this integral in terms of the function Φ^*.

Consider the representation π_Λ of G on $L_2(K)$ given by

$$\pi_\Lambda(x)f(u) = e^{-(\Lambda+2\rho)(H(x^{-1},u))}f(u_{x^{-1}}) \qquad (x\in G,\ u\in K,\ f\in L_2(K))$$

in the notation of [6, §12]. Let $\mathfrak{H}$ be the smallest closed subspace of $L_2(K)$ which is invariant under $\pi_\Lambda(G)$ and which contains the constant function 1. Then we have seen in [5, Part IV] that the representation π of G induced on $\mathfrak{H}$ is quasi-simple and its infinitesimal character(12) is χ_Λ where Λ is to be extended to a linear function on $\mathfrak{h}$ by putting it equal to zero on $\mathfrak{h}_{\mathfrak{k}}$. Let ψ_0 denote the vector in $\mathfrak{H}$ corresponding to the constant function 1. Then if we denote the scalar product of two elements in the usual way we get

$$(\psi_0, \pi(x)\psi_0) = \int_K e^{\Lambda(H(x,u))}du.$$

On the other hand suppose $x=\exp tH_0$ where $H_0\in\mathfrak{h}_{\mathfrak{p}_0}$ and $t\in R$. Then since ψ_0 is well-behaved under π (see Lemma 34 of [6]),

$$(\psi_0, \pi(\exp tH_0)\psi_0) = \sum_{m\geq 0} \frac{t^m}{m!}(\psi_0, \pi(H_0^m)\psi_0)$$

provided $|t|$ is sufficiently small. Now put $i_+(X)=(X+(-1)^{1/2}i(X))/2$, $i_-(X)=(X-(-1)^{1/2}i(X))/2$ ($X\in\mathfrak{k}$). Then $\mathfrak{k}_+=i_+(\mathfrak{k})$, $\mathfrak{k}_-=i_-(\mathfrak{k})$ are ideals in $\mathfrak{g}$ and $\mathfrak{g}$ is their direct sum. Let $\mathfrak{X}_+$ and $\mathfrak{X}_-$ be the subalgebras of $\mathfrak{B}$ generated by $(\mathfrak{k}_+, 1)$ and $(\mathfrak{k}_-, 1)$ respectively. Then if $a\in\mathfrak{X}_+$ and $b\in\mathfrak{X}$, $ab-ba=0$.

(12) This is easily seen by the argument used in the proof of Lemma 48 of [5].

Choose $H_0^* \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $H_0 = i(H_0^*)$. Then

$$H_0 = i(H_0^*) = 2(-1)^{1/2}[i_-(H_0^*) - H_0^*].$$

Moreover H_0^* and $i_-(H_0^*)$ commute and $\pi(\mathfrak{k})\psi_0 = \{0\}$. Hence

$$\pi(H_0^m)\phi_0 = (2(-1)^{1/2})^m \pi(H_-^m)\psi_0$$

where $H_- = i_-(H_0^*)$. On the other hand if $X, Y \in \mathfrak{k}$,

$$[X, i_-(Y)] = [i_-(X), i_-(Y)].$$

Hence $[X, z] = [i_-(X), z]$ $(X \in \mathfrak{k}, z \in \mathfrak{X}_-)$. Therefore we get a representation ν of $\mathfrak{k}$ on $\mathfrak{X}_-$ such that $\nu(X)z = [X, z]$ $(X \in \mathfrak{k}, z \in \mathfrak{X}_-)$. Obviously ν is quasi semisimple (see Lemma 10 of [6]). Therefore if z is any element in $\mathfrak{X}_-$, it follows from Lemma 7 of [6] that $z \equiv z_0 \bmod \nu(\mathfrak{k})\mathfrak{X}_-$ where z_0 is some element of $\mathfrak{X}_-$ which commutes with $\mathfrak{k}$. But then

$$[i_-(X), z_0] = [X, z_0] = 0 \qquad (X \in \mathfrak{k})$$

and moreover z_0 commutes with $\mathfrak{k}_+$ since it lies in $\mathfrak{X}_-$. Therefore z_0 is in the center of $\mathfrak{B}$. Furthermore the representation of K induced under π is unitary and therefore if $X \in \mathfrak{k}_0$ and $a \in \mathfrak{B}$,

$$(\psi_0, \pi(Xa - aX)\psi_0) = (-\pi(X)\psi_0, \pi(a)\psi_0) = 0$$

since $\pi(\mathfrak{k})\psi_0 = \{0\}$. This shows that

$$(\psi_0, \pi(z)\psi_0) = (\psi_0, \pi(z_0)\psi_0) = \chi_\Lambda(z_0).$$

Now if we extend χ_Λ to a linear function on $\mathfrak{B}$ such that $\chi_\Lambda(ab) = \chi_\Lambda(ba)$ $(a, b \in \mathfrak{B})$ (see Part III of [5]) it follows that $\chi_\Lambda(z_0) = \chi_\Lambda(z)$. Hence

$$(\psi_0, \pi(z)\psi_0) = \chi_\Lambda(z) \qquad (z \in \mathfrak{X}_-).$$

This proves that

$$(\psi_0, \pi(\exp tH)\psi_0) = \sum_{m \geq 0} \frac{t^m}{m!} (2(-1)^{1/2})^m \chi_\Lambda(H_-^m).$$

We extend i_- to an isomorphism of $\mathfrak{X}$ with $\mathfrak{X}_-$. Then the mapping $z \rightarrow \chi_\Lambda(i_-(z))$ $(z \in \mathfrak{X})$ is clearly a character of the algebra $\mathfrak{X}$. Hence from Theorem 5 of [5] there exists a linear function λ on $\mathfrak{h}_{\mathfrak{k}}$ such that $\chi_\Lambda(i_-(z)) = \xi_\lambda(z)$ $(z \in \mathfrak{X})$. Therefore

$$(\psi_0, \pi(\exp tH_0)\psi_0) = \sum_{m \geq 0} \frac{t^m}{m!} (2(-1)^{1/2})^m \xi_\lambda(H_0^{*m}) = \Phi^*(\lambda, 2(-1)^{1/2} tH_0^*)$$

if $|t|$ is sufficiently small. In view of equation (25) (p. 81) of [5], λ may be chosen in such a way that $\lambda(H) = \Lambda(i_-(H))$ $(H \in \mathfrak{h}_{\mathfrak{k}})$. Since Λ vanishes on $\mathfrak{h}_{\mathfrak{k}}$ we conclude that $\lambda(H) = -((-1)^{1/2}/2)\Lambda(i(H))$ and therefore $2(-1)^{1/2}\lambda(H_0^*) = \Lambda(H_0)$. Put $si(H) = i(sH)$ $(s \in W)$ and $\alpha(i(H)) = (-1)^{1/2}\alpha(H)$, $\sigma(i(H))$

$=(-1)^{1/2}\sigma(H)$ $(H\in\mathfrak{h}_{\mathfrak{k}})$. Then $\lambda(H_\alpha)=-((-1)^{1/2}/2)\Lambda(i(H_\alpha))=\Lambda(H_\alpha')$ where $H_\alpha'=-((-1)^{1/2}/2)i(H_\alpha)$ and $\sigma(H_\alpha')=\sigma(H_\alpha)/2$. Then if we put

$$\Phi(\Lambda, H) = \frac{\prod_{\alpha>0} 2\sigma(H_\alpha')}{\prod_{\alpha>0}(\Lambda+2\sigma)(H_\alpha')}\;\frac{\sum_{s\in W}\epsilon(s)e^{(\Lambda+2\sigma)(sH)}}{\prod_{\alpha>0}(e^{\alpha(H)}-e^{-\alpha(H)})}\qquad (H\in\mathfrak{h}_{\mathfrak{p}_0})$$

it is clear that $\Phi(\Lambda, H)$ is an analytic function on $\mathfrak{h}_{\mathfrak{p}_0}$ and

$$\Phi^*(\lambda, 2(-1)^{1/2}tH_0^*) = \Phi(\Lambda, tH_0).$$

Hence $(\psi_0, \pi(\exp tH_0)\psi_0)=\Phi(\Lambda, tH_0)$ for all sufficiently small values of $|t|$. Since both sides are analytic functions of t, the equality must hold for all values of t. Thus we have the following result.

THEOREM 7. *Let Λ be a linear function on $\mathfrak{h}_{\mathfrak{p}}$. Then if $x=\exp H$ $(H\in\mathfrak{h}_{\mathfrak{p}_0})$ we have the formula*

$$\int_K e^{\Lambda(H(x,u))}du = \frac{\prod_{\alpha>0} 2\sigma(H_\alpha')}{\prod_{\alpha>0}(\Lambda+2\sigma)(H_\alpha')}\;\frac{\sum_{s\in W}\epsilon(s)e^{(\Lambda+2\sigma)(sH)}}{\prod_{\alpha>0}(e^{\alpha(H)}-e^{-\alpha(H)})}.$$

Put $\phi(x)=\int_K e^{\Lambda(H(x,u))}du=(\psi_0, \pi(x)\psi_0)$ $(x\in G)$. Then it is clear that $\phi(uxv)=\phi(x)$ $(u, v\in K)$. Since every element in G can be written in the form $u(\exp H)v$ $(H\in\mathfrak{h}_{\mathfrak{p}_0};\ u, v\in K)$, the above formula determines ϕ completely.

The particular case of this formula for the complex immodular group has been obtained by Gelfand and Naimark [2, p. 77] by means of a lengthy computation (see also [1]).

REFERENCES

1. Gelfand and Naimark, Doklady Nauk SSR (N.S.) vol. 63 (1948) pp. 225–228.
2. ———, Trudi Mat. Inst. Steklova vol. 36 (1950).
3. R. Godement, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 496–556.
4. Harish-Chandra, Ann. of Math. vol. 51 (1950) pp. 299–330.
5. ———, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 28–96.
6. ———, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 185–243.
7. ———, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 26–65.
8. ———, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) (a) pp. 170–173; (b) pp. 362–365; (c) pp. 366–369; (d) pp. 691–694.
9. L. Schwartz, *Théorie des distributions*, Paris, Hermann, 1950.
10. I. E. Segal, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 13–15.

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